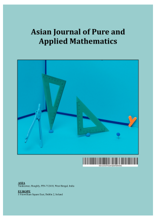

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Some Remarks on Almost Similarity and Other New Equivalence Relations of Operators

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, new results touching on almost similarity and other equivalence relations are presented. We investigate the relationship between almost-similarity relation and other equivalence relations of bounded linear operators on Hilbert spaces. We study the relation between equivalence classes of bounded linear operators with respect to different properties such as being self-adjoint, projections, normal, unitary and

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having a specific rank. This paper improves on previous works on known operator equivalence relations and classes. We obtain several new results on operator equivalences. Methodology involved the use of properties of orthogonal operators and normality of operators among others. Results show that almost similar projection operators have the same spectrum. Results touching on spectral invariants with respect to some of these equivalence relations are also established.

Keywords: *Similar; almost similar; metrically equivalent; almost unitary equivalence; unitary-quasi-equivalence.*

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1 Introduction

Let $\mathcal{H}$ denote a Hilbert space and $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators. If $T \in B(\mathcal{H})$, then T^* denotes the adjoint of T, while $Ker(T)$, $Ran(T)$, $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ stands for the kernel of T, range of T, closure of $\mathcal{M}$ and orthogonal complement of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. We denote by $\sigma(T)$, $\|T\|$ and $W(T)$, the spectrum, norm and numerical range of T, respectively.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *similar* (denoted $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are `unitarily equivalent` (denoted by $A \simeq B$) if there exists a unitary operator $U \in B_+(\mathcal{H}, \mathcal{K})$ (Banach algebra of all invertible operators in $B(\mathcal{H})$) such that $UA = BU$ (i.e. $A = U^*BU$, equivalently, $A = U^{-1}BU$). In this case, we can then view A and B as abstractly the same operator. Any (basis-free) geometric or algebraic property of A is automatically inherited by B. Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be `metrically equivalent` (denoted by $A \overset{\mathtt{m.e}}{\sim} B$) if $\|Ax\| = \|Bx\|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in \mathcal{H}$ or $A \overset{\mathtt{m.e}}{\sim} B$ if $A^*A = B^*B$. (cf. [1]).

Two linear bounded linear operators A and B are said to be `nearly-equivalent`(denoted by $A \overset{\mathtt{n.e}}{\sim} B$) if A^*A and B^*B are similar and are `unitarily-quasi-equivalent` (denoted by $A \overset{\mathtt{u.q.e}}{\simeq} B$) if there is a unitary operator U such that $A^*A = UB^*BU^*$(see [2]). An operator T is said to be *nearly normal* if $T^*T = ATT^*A^{-1}$, where A is an invertible operator (cf. [3]). Two operators A and B in $B(\mathcal{H})$ are said to be `almost similar` (denoted by $A \overset{a.s}{\sim} B$) if there is an invertible operator N such that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$ (cf. [4],[5]). Two operators A and B in $B(\mathcal{H})$ are said to be `almost unitarily equivalent` (denoted by $A \overset{a.u.e}{\sim} B$) if there is a unitary operator U such that $A^*A = U^*(B^*B)U$ and $A^* + A = U^*(B^* + B)U$(cf. [6]). The proofs that unitary equivalence, similarity, quasi-similarity, metric equivalence and almost similarity are equivalence relations on $B(\mathcal{H})$ have appeared in [5],[7] and [8].

Remark. Note that two operators are said to be equivalent if they are "close" to each other in some sense. Two operators are almost similar if the squares of their absolute values are similar and their real parts are similar.

An operator $T \in B(\mathcal{H})$ is self-adjoint if $T^* = T$; an isometry if $T^*T = I$; a co-isometry if $TT^* = I$, a partial isometry if $TT^*T = T$ or if $T^*T = P$, where P is an orthogonal projection; unitary if $T^*T = TT^* = I$; normal if $T^*T = TT^*$; quasinormal if $T(T^*T) = (T^*T)T$(see [9], [10], [11], [12], [13]).

Recently, a number of authors (see [7], [8]) have continued to characterize this relation and closely related relations of operators. This has led to a deep understanding of the spectral properties of operators contained in these classes.

To set the stage, we first need some basic facts.

2 Preliminary Results

Proposition 2.1 ([13], Proposition 1.5). *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and B is hermitian, then A is hermitian.*

It is clear that unitary equivalence and almost unitary equivalence of operators imply almost similarity of operators.

Proposition 2.2 ([1], Corollary 2.27). *If $A, B \in B(\mathcal{H})$ are normal operators which are similar, then they are unitarily equivalent.*

Proposition 2.3 ([5], Proposition 1.3). *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and if A is hermitian, then A and B are unitarily equivalent.*

Proof. By assumption there exists an invertible operator N such that $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$. Since A is hermitian and
$A \overset{a.s}{\sim} B$, by Proposition 2.1 it follows that B is hermitian. Using this fact, the second equality above becomes $A = N^{-1}QN$. This means that A and B are similar. We have now shown that these operators are both hermitian and are therefore normal. The claim follows by application of Proposition 2.2.

Corollary 2.1. *If $A, B \in B(\mathcal{H})$ such that $A \overset{a.s}{\sim} B$ and if either A or B is hermitian, then A and B are unitarily equivalent.*

Proof. The proof follows easily from Proposition 2.3.

Proposition 2.4. *Let A and B be bounded linear operators on a finite dimensional Hilbert space $\mathcal{H}$. Then A is similar to B if and only if $dim(Ker(A - \lambda I)^k) = dim(Ker(B - \lambda I)^k)$ for all $\lambda \in \mathbb{C}, k \geq 1$.*

Proposition 2.5 says that A is similar to B if and only if A and B have the same eigenvalues with the same algebraic and geometric multiplicities.

Remark. Almost similarity of operators $A, B \in B(\mathcal{H})$ can be defined in terms of squares of their absolute values and their real Cartesian parts. Two operators are almost similar if the squares of their absolute values are similar and their real parts are similar. That is, two operators A and B in $B(\mathcal{H})$ are said to be `almost similar` (denoted by $A \overset{a.s}{\sim} B$) if there is an invertible operator N such that $|A|^2 = N^{-1}(|B|^2)N$ and $Re(A) = N^{-1}(Re(B))N$.

Clearly, unitary equivalence implies almost similarity on $B(\mathcal{H})$.

3 Main Results

Theorem 3.1. *Almost similarity, near equivalence and unitary-quasi-equivalence relations are equivalence relations on $B(\mathcal{H})$.*

Proof. The proof that almost similarity and unitary-quasi-equivalence are equivalence relations follows from proof of ([5], Theorem 2.1 and [2], Theorem 2.1). The proof that near equivalence is an equivalence relation follows from the definition and that of unitary-quasi-equivalence.

Theorem 3.2. *Let P and Q be orthogonal projection operators on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent.*

(a). P and Q are almost similar.

(b). P and Q are similar.

(c). P and Q are unitarily equivalent.

Proof. (a)$\Rightarrow$ (b): Suppose that N is an invertible operator such that $P^*P = N^{-1}(Q^*Q)N$ and $P^* + P = N^{-1}(Q^* + Q)N$. Since P and Q are orthogonal projections, a simple computations shows that these two equalities both collapse to the equality $P = N^{-1}QN$.

(b)$\Rightarrow$ (a): Suppose $P = N^{-1}QN$ for some invertible operator $N \in B(\mathcal{H})$ and orthogonal projections P and Q in $B(\mathcal{H})$. By the idempotence of P and Q we have that $P^2 = N^{-1}Q^2N$. By the self-adjointness of P and Q, we also have $P^*P = N^{-1}(Q^*Q)N$. The other equality follows from the fact that $P^* + P = 2P = 2N^{-1}(Q^* + Q)N$, which upon simplification becomes $P^* + P = N^{-1}(Q^* + Q)N$.

(b)$\Rightarrow$ (c): follows by the fact that similar normal operators are unitarily equivalent.

(c)$\Rightarrow$ (a):is trivial.

Remark. Theorem 3.2 says that for orthogonal projection operators of the same rank on a Hilbert space $\mathcal{H}$, the notion of almost similarity coincides with that of similarity and that of unitary equivalence. This result may fail for other classes of operators.

Theorem 3.3. *Almost unitary equivalence implies almost similarity on $B(\mathcal{H})$.*

Proof. The proof follows from the definition of both relations.

Theorem 3.4. *If $A, B \in B(\mathcal{H})$ are similar normal operators, then they are almost similar.*

Proof. Suppose $B = X^{-1}AX$. Since A and B are similar normal operators, by Putnam-Fuglede Theorem([14], Theorem 1 and [15], p. 315, [12], [16]), they are unitarily equivalent. That is, there is a unitary operator U such that $B = U^*AU$. A simple calculation shows $A^*A = U^*(B^*B)U$ and $A^* + A = U^*(B^* + B)U$and, that is, A and B are almost unitarily equivalent and hence almost similar by Theorem 3.3.

Remark. Note that similarity of A and B need not imply similarity of A^*A and B^*B or metric equivalence of A and B (unless both A and B are normal operators). Thus similarity of any two operators need not imply almost similarity or metric equivalence of the operators. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Then A and B are similar non-normal operators, with $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$ implementing the similarity. However, a simple calculation shows that $A^*A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = B^*B$. It is also clear from the computation that A^*A and B^*B are not similar and therefore A and B cannot be almost similar.

Similarity preserves those properties of operators which are algebraic in nature: spectra, multiplicity of eigenvalues and nullity. Note that if A and B are similar, then $dim(Ker(A)) = dim(Ker(B))$. This does not necessarily mean that they have equal kernels. The only properties of operators that survive almost similarity conjugation are: multiplicity, etc.

$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ are similar but $Ker(A) \neq Ker(B)$. However, a simple calculation shows they have the same nullity.

Question. Does almost similarity preserve spectra of operators?

We show that the only properties of operators that survive almost similarity conjugation are multiplicity, etc. Spectra is not invariant under this relation.

The following result gives conditions under which almost similarity preserves spectra.

Theorem 3.5. *If P and Q are almost similar projections, then they have the same spectrum.*

Proof. By Theorem 3.2, P and Q are similar and hence have the same spectrum.
This also follows from the fact that similar normal operators are unitarily equivalent.

Theorem 3.5 can be extended to almost similar self-adjoint operators.

Proposition 3.1. *If A and B are self-adjoint operators which are almost similar then $\sigma(A) = \sigma(B)$.*

Proof. Suppose $A^* = A, B^* = B$ and $A^*A = N^{-1}(B^*B)N$ and $A^* + A = N^{-1}(B^* + B)N$. A simple calculation using the second condition shows that $A = N^{-1}B\text{N}$. That is A and B are similar and therefore have equal spectrum.

Theorem 3.6. *If T and S are almost similar, then so are T^* and S^*.*

Corollary 3.7. *If T and S are almost similar, then so are T^*T and S^*S.*

Corollary 3.8. *If T and S are almost similar, then so are the defect operators $D_T = \sqrt{I - T^*T}$ and $D_S = \sqrt{I - S^*S}$.*

The above result also tells us that if two operators are almost similar, then they have the same index. The converse is, however not true. The unilateral shift S and the identity operator I on $\ell^2(\mathbb{N})$ have index 0. That is $d_S = d_I = 0$, but these two operators are not almost similar.

Theorem 3.9. *If T and S are unitarily equivalent operators, then so are T^n and S^n, for any positive integer n.*

Proof. Unitary equivalence of T and S implies the existence of a unitary operator U such that $T = U^*SU$. A simple computation shows that $T^n = U^*S^nU$.

Corollary 3.10. *If T and S are unitarily equivalent normal operators, then so are T^n and S^n, for any positive integer n.*

Proof. The proof follows the normality of T and S and an application of Theorem 3.10.

Remark. Note that if A and B are almost similar with an invertible operator X implementing the almost similarity, then by mathematical induction on $n \in \mathbb{N}$ we have that $(A^*A)^n = N^{-1}((B^*B)^n)N$ and $(A^* + A)^n = N^{-1}((B^* + B)^n)N$ (cf. [17]).

Corollary 3.11. *If T and S are unitarily equivalent normal operators, then T^n and S^n are almost similar, for any positive integer n.*

Proof. Normality together with unitary equivalence of T and S and mathematical induction on $n \in \mathbb{N}$ imply that $T^{*n}T^n = (T^*T)^n = U^*((S^*S)^n)U = U^*(S^{*n}S^n)U$ and $T^{*n} + T^n = U^*(S^{*n} + S^n)U$.This proves that T^n and S^n are almost unitarily equivalent and hence almost similar by Theorem 3.3.

Question. Does almost similarity of S and T imply almost similarity of S^*S and T^*T?

Theorem 3.12. *Almost similarity of $S, T \in B(\mathcal{H})$ implies almost similarity of S^*S and T^*T.*

Proof. Let $\Omega = S^*S$ and $\Lambda = T^*T$. Almost similarity of S and T implies the existence of an invertible operator N such that $\Omega = N^{-1}\Lambda N$ and $S^* + S = N^{-1}(T^* + T)N$. A simple computation shows that $\Omega^*\Omega = N^{-1}(\Lambda^*\Lambda)N$ and $\Omega^* + \Omega = N^{-1}(\Lambda^* + \Lambda)N$. This proves the claim.

We note that almost similarity of S^*S and T^*T does not necessarily imply almost similarity of S and T. A good example is the unilateral shift S and the identity operator T on $\ell^2(\mathbb{N})$, which are not almost similar although S^*S and T^*T are almost similar.

Question. When does almost similarity of S^*S and T^*T imply almost similarity of S and T?

Theorem 3.13. *S^*S is almost similar to T^*T if and only if S and T are almost similar projections.*

Corollary 3.14. *S^*S is almost similar to T^*T if and only if S and T are similar projections.*

Proof. The proof follows from Theorem 3.14 and Theorem 3.2.

Example. The operators $A = diag(1, 1, 0)$ and $B = diag(1, 0, 1)$ are similar projections, with similarity isomorphism $N = \begin{pmatrix} 2 & -3 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$. It is easily verified that $A^*A = A$ and $B^*B = B$ are similar operators.

An operator $T \in B(\mathcal{H})$ is said to be `positive` if it is self-adjoint and $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. An operator $T \in B(\mathcal{H})$ is said to be `quasi-unitary` if $T^*T = TT^* = T^* + T$. The class of quasi-unitary operators was introduced by Phadke etal **[18]** . The class of unitary operators and that of quasi-unitary operators are disjoint subclasses of the class of normal operators. A simple calculation shows that if T is quasi-unitary, then $\sigma(T) \subseteq \{0, 2\}$. By definition, this class does not contain the identity operator. Indeed, every quasi-unitary operator is similar to a diagonal operator with either $0, 2$ or a combination of both on the main diagonal. Indeed, an operator T is quasi-unitary if it is normal and that $T^*T = T^* + T$. An operator T is called a θ-**operator** if T^*T and $T^* + T$ commute. Clearly, every quasi-unitary is a θ- operator. The class of θ- operators was introduced by Campbell**[19]**.

Proposition 3.2. *Every quasi-unitary operator is a positive real scalar multiple of a non-trivial projection operator.*

Theorem 3.15. *$T \in B(\mathcal{H})$ is quasi-unitary if and only if $I - T$ is unitary.*

Proof. Suppose T is quasi-unitary. Then $(I - T)^*(I - T) = (I - T)(I - T)^* = I$. Therefore $I - T$ is unitary. Conversely, suppose $I - T$ is unitary, then

$$I - (T + T^*) + T^*T = I - (T^* + T) + TT^* = I.$$

Simplifying the equation, we have that $T^*T = TT^* = T + T^*$. This establishes the claim.

Proposition 3.3. *If T is a quasi-unitary operator then T^* is also quasi-unitary.*

Theorem 3.16 ([20], Theorem C)**.** *If T is a θ- operator, then there exists some normal operator N such that $T^*T = N^*N$ and $T + T^* = N + N^*$.*

Corollary 3.17. *Every θ- operator is almost similar to a normal operator.*

Proof. The proof follows from Theorem 3.19.

Corollary 3.18. *If A and B are almost similar quasi-unitary operators then A^* and B^* are almost similar quasi-unitary operators.*

Proof. Since A and B are quasi-unitary, by Theorem 3.7 and Proposition 3.18, $A \overset{a.s}{\sim} A^*$ and $B \overset{a.s}{\sim} B^*$. Therefore $A \overset{a.s}{\sim} B$ implies that $A^* \overset{a.s}{\sim} B^*$. The rest of the claim follows from the fact that the adjoint of a quasi-unitary operator is also a quasi-unitary operator.

There exist non-normal similar operators $A, B \in B(\mathcal{H})$ that are not almost similar but A^*A and B^*B are almost similar.
Example. The operators $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are non-normal, similar partial isometries on $\mathcal{H} = \mathbb{R}^2$ which are not almost similar, with $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ implementing the similarity of A and B and $Y = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ implementing the almost-similarity of A^*A and B^*B.

Remark. The class of operators $\Omega_a = \{T \in B(\mathcal{H}) : T \overset{a.s}{\sim} T^*\}$ for which T and T^* are almost similar is more general than the class $\mathcal{N} = \{T \in B(\mathcal{H}) : T^*T = TT^*\}$ of normal operators. It contains all partial isometries and invertible quasinormal operators. However, the class $\mathcal{M}_e = \{T \in B(\mathcal{H}) : T^*T = TT^*\}$ of operators for which T and T^* are metrically equivalent coincides with the class of normal operators.

Theorem 3.19 ([20], Theorem C)**.** *If $T \in B(\mathcal{H})$ is a θ- operator, then there exists some normal operator N such that $T^*T = N^*N$ and $T + T^* = N + N^*$.*

Corollary 3.20. *Every θ- operator $T \in B(\mathcal{H})$ is almost similar to a normal operator.*

Proof. The proof follows from Theorem 3.22.

From Theorem 3.22, every θ- operator is metrically equivalent to and hence near equivalent to a normal operator. Let $T = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. A simple calculation shows that T is normal and that $T^* + T = 0$. Thus T is a θ- operator.

Remark. Note that for all $x \in \mathcal{H}$, $\||T|x\| = \|Tx\|$.

Proof. $\Big\| \, |T|x \Big\|^2 = \langle |T|x, |T|x\rangle = \langle |T|^2 x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$.

Remark. We note that "unitary equivalence to an isometry" is the same as "metric equivalence to a unitary".
Proof. Suppose A is an isometry and $B = U^*AU$. Then $B^*B = U^*A^*AU = U^*U = I$.

3.1 Metric, similarity and almost-similarity orbits

The `metric orbit` of an operator $T \in B(\mathcal{H})$ denoted by $\mathfrak{M}_e(T)$ is the class of operators metrically equivalent to T. That is, $\mathfrak{M}_e(T) = \{B \in B(\mathcal{H}) : B^*B = T^*T\} = \{B \in B(\mathcal{H}) : \|Bx\| = \|Tx\|, \ for\ all\ x \in \mathcal{H}\}$.

Clearly, $\mathfrak{M}_e(T)$ is a non-empty set, since it contains T. It is also clear that the metric orbit of an isometry T consists of isometries S such that $Re(S)$ and $Re(T)$ are similar.

The following two results are immediate consequences of the definition of metric orbit.

Theorem 3.21. *Let $A, B \in B(\mathcal{H})$. Then $B \in \mathfrak{M}_e(A)$ if and only if $B = VA$, for some isometry V.*

Proof. Suppose $B \in \mathfrak{M}_e(A)$. Then $B^*B = A^*A = A^*(V^*V)A = (VA)^*(VA)$. This implies that $B = VA$, where V is an isometry. Conversely, suppose $B = VA$, for some isometry V. Then $B^*B = (VA)^*(VA) = A^*(V^*V)A = A^*A$ and therefore $B \in \mathfrak{M}_e(A)$.

We note that the isometry V in Theorem 3.24 need not be invertible.

Theorem 3.22. *Let $A, B \in B(\mathcal{H})$. Then A and B are metrically equivalent if and only if $\mathfrak{M}_e(A) = \mathfrak{M}_e(B)$.*

Proof. If A and B are metrically equivalent then by definition

$$\mathfrak{M}_e(A) = \{T \in B(\mathcal{H}) : T^*T = A^*A\} = \{T \in B(\mathcal{H}) : T^*T = B^*B\} = \mathfrak{M}_e(B).$$

Conversely, suppose $\mathfrak{M}_e(A) = \mathfrak{M}_e(B)$. Then for every $E \in \mathfrak{M}_e(A)$ there is an $F \in \mathfrak{M}_e(B)$ such that $E^*E = F^*F$ and vice versa. But $A \in \mathfrak{M}_e(A)$ and $B \in \mathfrak{M}_e(B)$. Since the $E \in \mathfrak{M}_e(A)$ was arbitrarily chosen, without loss of generality, we let $E = A$ and $F = B$. This proves the claim.

Example 3.23.

Let $A = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}$ on $\mathcal{H} = \mathbb{C}^2$. Then a simple computation shows that $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\mathfrak{M}_e(A)$ if and only if $a = c = 0$ and $|b|^2 + |d|^2 = |\alpha|^2$. Therefore

$$\mathfrak{M}_e(A) = \Big\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} : |b|^2 + |d|^2 = |\alpha|^2 \Big\}.$$

In particular, for $\alpha = 1$:

$$\mathfrak{M}_e(A) = \Big\{ \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & \pm i \end{pmatrix}, \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} \\ 0 & \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}}i \\ 0 & \pm\frac{1}{\sqrt{2}}i \end{pmatrix},$$

$$\begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}}i \\ 0 & \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 & \pm\frac{1}{\sqrt{2}} \\ 0 & \pm\frac{1}{\sqrt{2}}i \end{pmatrix}, \begin{pmatrix} 0 & \pm(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \pm(\frac{1}{2} + \frac{1}{2}i) \\ 0 & \pm(\frac{1}{2} + \frac{1}{2}i) \end{pmatrix}, \ldots \Big\}.$$

Question 1. *Which classes of operators is similarity, almost-similarity the same as metric equivalence?*

To answer this question, we introduce the `similarity` and `almost-similarity` orbits $\mathfrak{S}_(T)$ and $\mathfrak{S}_a(T)$, of a bounded linear operator T, respectively:

$$\mathfrak{S}(T) = \{B \in B(\mathcal{H}) : B = X^{-1}TX,\ X \in B(\mathcal{H})\ invertible\}$$

$$\mathfrak{S}_a(T) = \{B \in B(\mathcal{H}) : B^*B = X^{-1}(T^*T)X \quad and \quad B^* + B = X^{-1}(T^* + T)X, X \in B(\mathcal{H})\ invertible\}.$$

We note that in general neither of these sets needs to be closed. We denote the corresponding (norm) closures of these sets as $\overline{\mathfrak{S}(T)}$ and $\overline{\mathfrak{S}_a(T)}$, respectively. In infinite dimensions $\mathfrak{S}(T)$ is closed if and only if T is similar to a normal matrix (see [21], Corollary 2.3).

Example 3.24.

For any $\epsilon > 0$, the operator $A = \begin{pmatrix} 0 & \epsilon \\ 0 & 0 \end{pmatrix}$ is similar to $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and clearly $T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \overline{\mathfrak{S}(B)}$, but $T \notin \mathfrak{S}(B)$. Note also that $A \in \mathfrak{M}_e(B)$ if and only if $\epsilon = 1$ and that $T \notin \overline{\mathfrak{M}_e(B)}$.

Remark. A closer look in Example 3.27 shows that the only operator similar and metrically equivalent to A is A itself. That is, $\mathfrak{M}_e(A) \cap \mathfrak{S}(A) = A$.

Proposition 3.4. *Let $P \in B(\mathcal{H})$ be an orthogonal projection, then*

(a). $\mathfrak{M}_e(P) \subseteq \mathfrak{S}(P)$.

(b). $\mathfrak{M}_e(P) \subseteq \mathfrak{S}_a(P)$.

(c). $\mathfrak{S}_a(P) = \mathfrak{S}(P)$.

Proposition 3.5. *Two operators A and B on a Hilbert space $\mathcal{H}$ are similar if and only if $\mathfrak{S}(A) = \mathfrak{S}(B)$.*

Proof. A and B similar implies existence of an invertible operator X such that $A = X^{-1}BX$. Thus

$$\begin{aligned} \mathfrak{S}(A) = \{F \in B(\mathcal{H}) : F = X^{-1}AX\} &= \{F \in B(\mathcal{H}) : F = X^{-1}(X^{-1}BX)X\} \\ &= \{F \in B(\mathcal{H}) : F = W^{-1}BW, \quad W = X^2\} \\ &= \mathfrak{S}(B). \end{aligned}$$

Conversely, suppose that $\mathfrak{S}(A) = \mathfrak{S}(B)$. Then by definition of set equality (Principle of Extensionality), every $E \in \mathfrak{S}(A)$ belongs to $\mathfrak{S}(B)$ and vice versa. That is, $E = X^{-1}FX$, for some $F \in \mathfrak{S}(B)$. But trivially, $A \in \mathfrak{S}(A)$ and $B \in \mathfrak{S}(B)$. Without loss of generality, letting $E = A$ and $F = B$ establishes the claim.

Proposition 3.6. *Two operators A and B are almost-similar if and only if $\mathfrak{S}_a(A) = \mathfrak{S}_a(B)$.*

Question 2. *What is the structure of $\mathfrak{S}_a(A)$ and $\mathfrak{M}_e(A)$ for a given $A \in B(\mathcal{H})$?*

Every $T \in B(\mathcal{H})$ admits a Cartesian decomposition as $T = Re(T) + iIm(T)$, where $Re(T) = \frac{T+T^*}{2}$ and $Im(T) = \frac{T-T^*}{2i}$ are the real and imaginary parts of T, respectively.

Lemma 3.25. *If $T \in B(\mathcal{H})$ is an isometry, then $\mathfrak{S}_a(T) = \{S \in B(\mathcal{H}) : S^*S = I$ and $Re(S) \sim Re(T)\}$.*

Proof. Let T be an isometry. Then
$\mathfrak{S}_a(T) = \{S \in B(\mathcal{H}) : S^*S = X^{-1}(T^*T)X \;\; \textit{and} \;\; S^* + S = X^{-1}(T^* + T)X, X \in B(\mathcal{H}) \textit{ invertible}\} = \{S \in B(\mathcal{H}) : S^*S = I \;\; \textit{and} \;\; S^* + S = X^{-1}(T^* + T)X, X \in B(\mathcal{H}) \textit{ invertible}\} = \{S \in B(\mathcal{H}) : S^*S = I \;\; \textit{and} \;\; 2Re(S) = X^{-1}(2Re(T))X, X \in B(\mathcal{H}) \textit{ invertible}\} = \{S \in B(\mathcal{H}) : S^*S = I \;\; \textit{and} \;\; Re(S) \sim Re(T)\}$.

Example. Let $T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ be an operator acting on $\mathcal{H} = \mathbb{C}^2$. Clearly T is an isometry and has a Cartesian decomposition as $T = Re(T) + iIm(T) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.
So, $\mathfrak{S}_a(T) = \{S \in B(\mathcal{H}) : S^*S = I \;\; \textit{and} \;\; Re(S) \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}$ That is, the almost similarity orbit of T consists of all isometries S acting on $\mathcal{H} = \mathbb{C}^2$ whose real parts are similar to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. An isometry like $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not belong to the almost similarity orbit of T since $Re(S) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = Re(T)$.

Lemma 3.26. *If $T \in B(\mathcal{H})$ is a self-adjoint isometry(ie unitary), then $\mathfrak{S}_a(T) = \{S \in B(\mathcal{H}) : S^*S = I$ and $Re(S) \sim T\}$.*

Proof. By the self-adjointness of T, we have that $Re(T) = T$. The rest of the proof follows from Lemma 3.31.

Example. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ be operators acting on $\mathcal{H} = \mathbb{C}^2$. A simple computation shows that A and B are unitarily equivalent (hence similar), metrically equivalent and almost unitarily equivalent(hence almost similar). So, in this case, $\mathfrak{S}(A) = \mathfrak{S}(B)$, $\mathfrak{M}_e(A) = \mathfrak{M}_e(B)$ and $\mathfrak{S}_a(A) = \mathfrak{S}_a(B)$, with the similarity and almost similarity being implemented by the unitary operator $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

It is easy to verify that for these two operators that

$$\begin{aligned} \mathfrak{S}_a(A) &= \{S \in B(\mathcal{H}) : S^*S \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } Re(S) \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\} \\ &= \{S \in B(\mathcal{H}) : T^*T \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } Re(T) \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\} \\ &= \mathfrak{S}_a(B). \end{aligned}$$

4 Discussion and Conclusion

The notion of equivalence(unitary equivalence, similarity, metric equivalence, almost similarity and so on) of operators in Hilbert spaces is applicable in ”equivalence checking” in the design of Boolean or digital circuits. This is the problem of checking/proving that two given combinational circuits F and G are functionally equivalent- that is checking whether two given circuit descriptions specify the same behaviour(cf. [22]). This takes the form of deciding if given two representations d_f and d_g of two Boolean functions acting on the Boolean algebra/Boolean n-space or ”switching algebra” $(\{0,1\}^n, \leq)$, that is $f, g : \{0,1\}^n \longrightarrow \{0,1\}^n$, whether the functions f and g are equal, that is, $f(\alpha) = g(\alpha)$ holds for all $\alpha \in \{0,1\}^n$. Equivalence can also be used to check for functionally equivalent states and functionally equivalent finite state machines(FSM)(cf. [22]).

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

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Competing Interests

Authors have declared that no competing interests exist.

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