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The Effects of Harvesting and Time Delay on Prey-predator Systems

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, we are interested in studying the stability of the equilibrium points of harvesting of a prey-predator system with time delay in the growth rate of the predator population. Firstly, we state the formulation of the model. Secondly, we drive different conditions stability of the equilibrium of the system, respectively. Constant effort harvesting of the prey has been incorporated in the model to cater for the effects of human poaching. Finally, we illustrate our results by some examples. The objective of this paper is to study the effects of harvesting and time delay on the dynamics of predator-prey system.

Keywords: Prey-predator model; time delay; harvesting; stability.

1 Introduction

Prey-predator interaction is the fundamental structure in population dynamics. Understanding the dynamics of predator-prey models is very helpful for investigating multiple species interactions. The prey-predator

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interaction has been described firstly by two pioneers Lotka [1] and Volterra [2] in two independent works. The Lotka-Volterra system is one of the most predator-prey systems to be based on sound mathematical principles. It forms the basis of many systems used today in the analysis of population dynamics. One of the models examined after Lotka-Volterra model is due to Clark [3]. In this model, prey-predator harvesting is applied on predator population as well as prey population, which grow logistically. Most of the existing prey-predator mathematical based models are developed and formulated considering the diversity of interest scenarios such as Holling Type functional responses [4], ratio-dependent functional responses [5,6], bio-economic exploitation or harvesting [7], and stage or age structured [8,9].

The subject of harvesting in predator-prey systems has been of interest to economists, ecologists and natural resource managers for some time now. Harvesting has a strong impact on the dynamic evaluation of a population subjected to it. Effects of harvesting on various types of prey-predator models have been considered by many researchers [10-15]. Mathematical modeling with harvesting renewable resources started with the studies of Clark [16,17] has investigated the dynamics of a system with constant harvesting on the predator. The problem of harvesting with time delay in the predator-prey system is an important for study. In general, delay differential equations are more complicated dynamics than ordinary differential equations as time delay could cause a stable equilibrium to become unstable.

In this paper, we present a deterministic and continuous model for prey-predator population based on Lotka-Volterra model which is extended by incorporating time delay in the growth rate of the prey population and constant rate of harvesting of the predator population. The constant rate of harvesting could put the predator-prey system to none, or at least positive equilibrium points. The objective of this paper is to study the effects of harvesting and time delay on the dynamics of predator-prey system.

The rest of this paper is organized as follows: Section 2 discusses a general description of a prey-predator system. In section 3, we extend the prey-predator system to include harvesting. In section 4, we integrate the concept of time delay in prey-predator system with harvesting. A brief concluding remark is given in Section 5.

2 A Prey-Predator System

Let in an eco-system there are only two types of animal namely: The prey and the predator. They from a simple food-chain where the prey species grazes vegetation, while the predator species hunts the prey species. The size of the two populations can be described by a simple system of two nonlinear first order differential equations (a.k.a. the Lotka-Volterra equations).

Consider $x(t)$ denotes the population of the prey species and $y(t)$ denotes the population of the predators population species at any time t. Then

$$\frac{dx(t)}{dt} = r_1 x - \alpha x y$$

$$\frac{dy(t)}{dt} = -r_2 y + \beta x y \tag{1}$$

where r_1 is the maximum rate of the prey population, r_2 is the relative rate at which the predators die out in absence of prey, α measures the rate consumption of prey, and β measures the conversion of prey consumed into the predators reproduction rate.

It is worth mentioning that in the absence of the predator (i.e., $y = 0$), the prey population would grow exponentially. However, if the preys are absence (i.e., $x = 0$), the predator population would decay exponentially to zero due to starvation. One obvious shortcoming of the basic prey-predator system is that the population of the prey species would grow unbounded, in the absence of predators.

Das et al. [10] modified the basic prey-predator system given (1) incorporating the effect of toxic substances in a two species Lotka-Volterra competitive system as,

$$\frac{dx(t)}{dt} = r_1 x\left(1 - \frac{x}{l}\right) - \alpha xy$$

$$\frac{dy(t)}{dt} = -r_2 y + \beta xy \tag{2}$$

The system includes parameter l is the prey population in the absence of the predators.

The equilibrium points of system (2) occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$.

i.e.,

$$x\left(r_1\left(1 - \frac{x}{l}\right) - \alpha y\right) = 0$$

$$y(-r_2 + \beta x) = 0 \tag{3}$$

Solving the equations of system (3), then we can arrive into the following cases:

Case 1. When $x = y = 0$, the first equilibrium point is $\beta_1 = (0, 0)$.

Case 2. When $x = l$ and $y = 0$, the second equilibrium point is $\beta_2 = (l, 0)$.

Case 3. If $x \neq 0$, then from the first equation in system (3), we have

$$r_1\left(1 - \frac{x}{l}\right) - \alpha y = 0 \tag{4}$$

Also if $y \neq 0$, then from the second equation in system (3), we have

$$-r_2 + \beta x = 0, \text{ or, } y = \frac{r_1}{\alpha\beta l}(\beta l - r_2) \tag{5}$$

Thus, the third equilibrium point is $B_3 = \left(\frac{r_2}{\beta}, \frac{r_1}{\alpha\beta l}(\beta l - r_2)\right)$

Let $f(x, y) = x\left(r_1\left(1 - \frac{x}{l}\right) - \alpha y\right)$, and $g(x, y) = y(-r_2 + \beta x)$. Then, the value of the Jacobian matrix of system (2) is

$$J(x, y) = \begin{pmatrix} r_1 - \frac{2r_1 x}{l} - \alpha y & -\alpha x \\ \beta y & -r_2 + \beta x \end{pmatrix} \tag{6}$$

Definition 1. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a function with continuous first derivatives. The Jacobian of the function f is the matrix J whose entries are given by $\partial f_i / \partial x_i$, where f_i is the ith entry in f and x_j is the jth independent variable.

Theorem 1. Suppose $f: \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear, with continuous first derivatives, and x_e is a critical point of the nonlinear system $x' = f(x)$.

1. If all eigenvalues of the Jacobian matrix $J(x_e)$ have negative real parts, then the critical point x_e is asymptotically stable.
2. If any eigenvalue of the Jacobian matrix $J(x_e)$ has a positive real part, then the critical point x_e is unstable.

So that, we arrive into the following Theorems to study the stability of equilibrium points of system (2).

Theorem 2. The equilibrium point β_1 of system (2) is asymptotically stable provided that $r_1 > 0$ and $r_2 > 0$.

Proof. The Jacobian matrix of system (2) as in equation (6).

Substituting the equilibrium point β_1 of system (2) into the Jacobian matrix of system (6). Then

$$J(\beta_1) = \begin{pmatrix} r_1 & 0 \\ 0 & -r_2 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at β_1 is $det(J(\beta_1) - \lambda I) = 0$.

Therefore, the eigenvalues are $\lambda_1 = r_1, \lambda_2 = -r_2$ as β_1 is a diagonal matrix.

From the conditions in Theorem 2, note that the eigenvalues are negative as in Theorem 1. This completes the proof.

Theorem 3. The equilibrium point β_2 of system (2) is asymptotically stable provided that $r_1 > 0$ and $r_2 > \beta l$.

Proof. The value of the Jacobian matrix of system (2) is given in equation (6).

Now, the value of the Jacobian matrix of system (2) at the equilibrium point β_2 of system (2) is

$$J(\beta_2) = \begin{pmatrix} -r_1 & \alpha l \\ 0 & -r_2 + \beta l \end{pmatrix}$$

Therefore, the characteristic equation of the Jacobian matrix at β_2 is $det(J(\beta_2) - \lambda I) = 0$.

Therefore, the eigenvalues are $\lambda_1 = -r_1, \lambda_2 = -r_2 + \beta l$ as β_2 is an upper matrix.

From the conditions in Theorem 3, we have that the eigenvalues are negative as in Theorem 1. This completes the proof.

Theorem 4. The equilibrium point β_3 of system (2) is asymptotically stable provided that $\beta < 0, l < 0$, $r_1 > 0$ and $r_2 > 0$.

Proof. The value of the Jacobian matrix of system (2) is given in equation (6). Thus, the value of the Jacobian matrix of system (2) at the equilibrium point β_3 of system (2) equal to

$$J(\beta_2) = \begin{pmatrix} r_1 - \frac{2r_1r_2}{\beta l} - (\beta l - r_2)\frac{r_1}{\beta l} & \frac{-\alpha r_2}{\beta} \\ (\beta l - r_2)\frac{r_1}{\alpha l} & 0 \end{pmatrix}$$

Thus, the characteristic equation of the Jacobian matrix at β_3 is $det(J(\beta_3) - \lambda I) = 0$. Therefore,

$$det(J(\beta_3) - \lambda I) = det\left(\begin{pmatrix} r_1 - \frac{2r_1r_2}{\beta l} - (\beta l - r_2)\frac{r_1}{\beta l} & \frac{-\alpha r_2}{\beta} \\ (\beta l - r_2)\frac{r_1}{\alpha l} & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \left(\lambda^2 + \frac{1}{\beta l}r_1r_2\lambda - \frac{1}{\beta l}r_1r_2^2 + r_1r_2\right) = 0$$

Let $s = \frac{1}{\beta l}r_1r_2$ and $k = -\frac{1}{\beta l}r_1r_2^2 + r_1r_2$, then the characteristic equation of the Jacobian matrix at β_3 is $\lambda^2 + s\lambda + k = 0$.

Then, the roots of the characteristic equation of the Jacobian matrix at β_3 are $\frac{-s\pm\sqrt{s^2-4k}}{2}$. From the conditions in the **Theorem 4**, note that the eigenvalues have negative real parts as in **Theorem 1**. This completes the proof.

Example 1. Consider system (2) with parameters $r_1 = -0.8$, $l = 100$, $\alpha = 0.2$, $r_2 = 0.05$, and $\beta = 0.1$. The equilibrium points of system (3) are $\beta_1 = (0,0)$, $\beta_2 = (100,0)$, and $\beta_3 = (0.5, -3.98)$. The eigenvalues for the Jacobian matrix at the equilibrium point β_1 are -0.8 and -0.05, since both are negative, therefore β_1 is asymptotically stable. The eigenvalues for the Jacobian matrix at the equilibrium point β_2 are 0.8 and 9.95, since both are positive, which implies that β_2 is unstable. The eigenvalues for the Jacobian matrix at the equilibrium point β_3 are -0.19751 and -0.20151, since one of them is positive, thus β_3 is unstable. The direction field for the equilibrium points of system (2) with the parameters is shown in the following Figs. 1, 2 and 3. From Figs. 1 and 2 we note that the equilibrium points β_2 and β_3 are both unstable, but the equilibrium point β_1 is asymptotically stable as in Fig. 3.

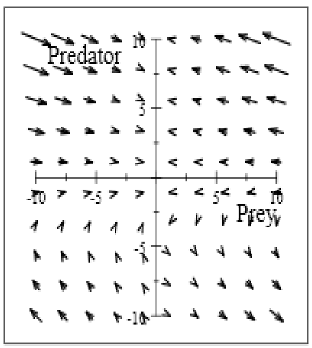

Fig. 1. Direction field for β_1

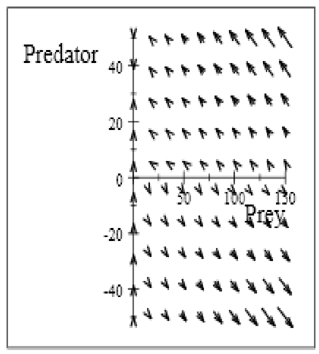

Fig. 2. Direction field for β_2

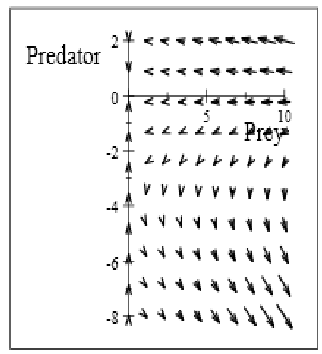

Fig. 3. Direction field for β_3

3 A Prey-Predator System with Constant Rate of Harvesting

The prey-predator systems with harvesting have received a great deal of attention for the last few decades. Let the rate of harvesting for predator population in system (2) constant. Then

$$\frac{dx(t)}{dt} = r_1 x(1 - bx) - \alpha xy$$

$$\frac{dy(t)}{dt} = -r_2 y + \beta xy - H_y \tag{7}$$

where $r_1, b = \frac{1}{l}, \alpha, r_2, and\ \beta$ are positive constants and H_y is nonnegative. The constant H_y is the rate of harvesting for the predator.

The equilibrium points of system (7) occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$,
i.e.,

$$r_1 x(1 - bx) - \alpha xy = 0$$

$$-r_2 y + \beta xy - H_y = 0 \tag{8}$$

Accordingly, the following cases can be obtained:

Case 1. From system (7), we have $r_1 x(1 - bx) - \alpha xy = 0$

If $r_1 \neq 0, x \neq 0, y = 0, H_y = 0$ in equation (3.3), then $r_1 x(1 - bx) = 0$

Thus, the equilibrium points is $A = \left(\frac{1}{b}, 0\right)$.

Case 2. From system (7), we have $-r_2 y + \beta xy = H_y$.

If $x = 0, y \neq 0, H_y \neq 0$ in equation (8), then $-r_2 y = H_y$ which implies that $y = \frac{H_y}{-r_2}$

Thus, the equilibrium points is $B = \left(0, \frac{H_y}{-r_2}\right)$.

Case 3. If $r_1 \neq 0$, say for example $x = \frac{1}{b}, y \neq 0, H_y \neq 0$ in equation (8) then $-r_2 y + \beta xy = H_y$, that is, $y = \frac{H_y}{-r_2+\beta x}$. Thus, the equilibrium points is $C = \left(\frac{1}{b}, \frac{H_y}{-r_{2+\beta\frac{1}{b}}}\right)$.

The Jacobian matrix for the system (7) is $J(x, y) = \begin{pmatrix} r_1 - 2r_1 bx - \alpha y & -\alpha x \\ \beta y & -r_2 + \beta x \end{pmatrix}$.

The value for $J(x, y)$ at $A = \left(\frac{1}{b}, 0\right)$ is $J(x, y) = \begin{pmatrix} r_1 - 2r_1 & -\alpha\frac{r_1}{br_1} \\ 0 & -r_2 + \beta\frac{r_1}{br_1} \end{pmatrix} = \begin{pmatrix} -r_1 & -\alpha\frac{1}{b} \\ 0 & -r_2 + \beta\frac{1}{b} \end{pmatrix}$.

The characteristic equation of the Jacobian matrix at A is $det(J(A) - \lambda I) = 0$.

Thus,

$$det(J(A) - \lambda I) = det\begin{pmatrix} r_1 - \lambda & -\alpha\frac{1}{b} \\ 0 & -r_2 + \beta\frac{1}{b} - \lambda \end{pmatrix} = \lambda^2 + n\lambda + m = 0 \quad (9)$$

where $n = \frac{1}{b}(-\beta + br_1 + br_2)$, and $m = r_1 r_2 - \frac{1}{b}\beta r_1$.

The root of characteristic equation (3.10) are $\lambda_{\pm} = \frac{-n \pm \sqrt{n^2 - 4m}}{2}$

The roots of equation (9) have negative real parts if $n > 0$, and $m > 0$.

Thus, we introduce the following theorem:

Theorem 5. The equilibrium point *B* of system (7) is asymptotically stable when the following conditions are satisfied $n > 0$, and $m > 0$.

The value for $J(x, y)$ at $B = \left(0, \frac{H_y}{-r_2}\right)$ is $(x, y) = \begin{pmatrix} r_1 - \alpha\frac{H_y}{-r_2} & 0 \\ \beta\frac{H_y}{-r_2} & -r_2 \end{pmatrix}$.

The characteristic equation of the Jacobian matrix at B is $det(J(B) - \lambda I) = 0$.

Thus,

$$det(J(A) - \lambda I) = det\begin{pmatrix} r_1 - \alpha\frac{H_y}{-r_2} - \lambda & 0 \\ -\beta\frac{H_y}{-r_2} & -r_2 - \lambda \end{pmatrix} = \lambda^2 + k_1\lambda + k_2 = 0 \quad (10)$$

where $k_1 = r_2 - r_1 r_2 - \frac{\alpha}{-r_2}H_y$, and $k_2 = -r_1 r_2 - \alpha H_y$.

The roots of characteristic equation (9) are $\lambda_{\pm} = \frac{-k_1 \pm \sqrt{{k_1}^2 - 4k_2}}{2}$.

The roots of equation (9) have negative real parts if $k_1 > 0$, and $k_2 > 0$.

Thus, we introduce the following theorem:

Theorem 6. The equilibrium point B of system (7) is asymptotically stable when the following conditions are satisfied if $k_1 > 0$, and $k_2 < 0$.

The value for $J(x, y)$ at $C = (x_1, y_1)$ is

$$J(C) = \begin{pmatrix} r_1 - 2bx_1 - \alpha y_1 & -\alpha x_1 \\ \beta y_1 & -r_2 + \beta x_1 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix at B is

$$det(J(C) - \lambda I) = 0$$

Thus,

$$det(J(C) - \lambda I) = det\begin{pmatrix} r_1 - 2bx_1 - \alpha y_1 - \lambda & -\alpha x_1 \\ \beta y_1 & -r_2 + \beta x_1 - \lambda \end{pmatrix} = \lambda^2 + s_1\lambda + s_2 = 0 \qquad (11)$$

where $s_1 = r_1 + r_2 - \beta y_1 + \alpha y_1$, and $s_2 = \beta r_1 x_1 + \alpha r_2 y_1 - 2b\beta x_1^2 - r_1 r_2 + 2bx_1 + 2br_2 x_1$.

The roots of equation (11) are $\lambda_{\pm} = \frac{-s_1 \pm \sqrt{{s_1}^2 - 4s_2}}{2}$.

Then equation (11) has negative parts if and only if $s_1 > 0$ and $s_2 < 0$. Therefore the equilibrium point C is asymptotically stable when $s_1 > 0$ and $s_2 < 0$. Thus, we introduce the following theorem.

Theorem 7. The equilibrium point C of system (7) is asymptotically stable when the following conditions are satisfied $s_1 > 0$ and $s_2 > 0$.

Example 2. Consider system (7) with parameters $r_1 = 1, b = 1, \alpha = 1, r_2 = 0.5, \beta = 1$, and $H_y = 1$. The equilibrium points of system (7) are $P_1 = (1, 0)$, $P_2 = (0, -2)$, and $P_3 = (1, 2)$. The eigenvalues for the Jacobian matrix at the equilibrium point P_1 are 0.5 and -1, note that one of them is positive, which implies that P_1 is unstable. The eigenvalues for the Jacobian matrix at the equilibrium point P_2 are 2.5811 and -0.58114, note that one of them is positive, which implies that P_2 is unstable. The eigenvalues for the Jacobian matrix at the equilibrium point P_3 are $-0.25 + 1.5612i$ and $-0.25 + 1.5612i$, which have negative real parts, thus P_3 is unstable. The direction field for the equilibrium points of system (7) with the parameters is shown in the following Figs. 4, 5 and 6.

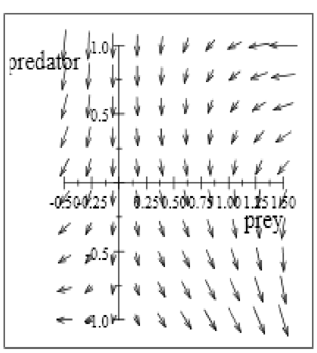

Fig. 4. Direction field for P_1

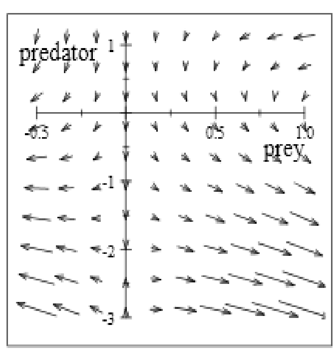

Fig. 5. Direction field for P_2

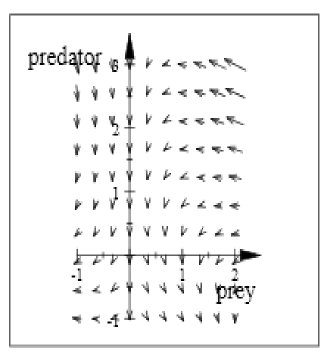

Fig. 6. Direction field for P_3

4 A Prey-Predator System with Time Delay

Now we consider the prey-predator system with time delay τ into the prey population in system (7) and constant rate of harvesting of predator population.

The system with time delay takes the form

$$\frac{\boldsymbol{dx(t)}}{\boldsymbol{dt}} = r_1x(t) - bx^2(t) - \alpha x(t)y(t-\tau)$$

$$\frac{dy(t)}{dt} = -r_2y(t) + \beta x(t)y(t) \tag{12}$$

Assume that $P(x_1, y_1)$ is a positive equilibrium point for system (12). Then, we study the stability of system (12) at P.

Now linearize system (12) about the equilibrium point P as follows:

Assume that $u(t) = x(t) - x_1$ and $v(t) = y(t) - x_1$, then $\frac{du}{dt} = \frac{dx}{dt}$, and $\frac{dv}{dt} = \frac{dy}{dt}$

Let $f(x,y) = r_1x(t) - bx^2(t) - \alpha x(t)y(t-\tau)$, $g(x,y) = -r_2y(t) + \beta x(t)y(t)$

Thus, the linearization for system (12) at $P(x_1, y_1)$ is

$$\frac{du}{dt} = \frac{\partial f}{\partial x}(P^*)u(t) + \frac{\partial f}{\partial y}(P^*)v(t-\tau)$$

$$\frac{du}{dt} = \frac{\partial g}{\partial x}(P^*)u(t) + \frac{\partial g}{\partial y}(P^*)v(t)$$

i.e., the linearization for system (4:1) is

$$\frac{du}{dt} = (r_1 - 2bx_1 - \alpha y_1)u(t) - \alpha x_1 v(t-\tau)$$

$$\frac{dv}{dt} = \beta y_1 u(t) + (-r_2 + \beta x_1)v(t) \tag{13}$$

The characteristic equation of system (13) is $\Delta(\lambda,\tau) = det\left(\lambda I - A_0 - \sum_{j=1}^{1} A_j e^{-\lambda\tau_j}\right)$.

where $A_0 = \begin{pmatrix} r_1 - 2bx_1 - \alpha y_1 & 0 \\ \beta y_1 & -r_2 + \beta x_1 \end{pmatrix}$ and $A_1 = \begin{pmatrix} 0 & -\alpha x_1 \\ 0 & 0 \end{pmatrix}$. Thus,

$$\Delta(\lambda,\tau) = det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} r_1 - 2bx_1 - \alpha y_1 & 0 \\ \beta y_1 & -r_2 + \beta x_1 \end{pmatrix} - \begin{pmatrix} 0 & -\alpha x_1 \\ 0 & 0 \end{pmatrix} e^{-\lambda\tau}\right) =$$
$$\begin{pmatrix} \lambda - r_1 + 2bx_1 + \alpha y_1 & \alpha x_1 e^{-\lambda\tau} \\ -\beta y_1 & \lambda + r_2 - \beta x_1 \end{pmatrix}$$

which implies that

$$\lambda^2 - a\lambda - be^{-\lambda\tau} + c = 0 \tag{14}$$

where $a = r_1 - r_2 - 2bx_1 + \beta x_1 - \alpha y_1$, $b = -\alpha\beta x_1 y_1$, and $c = -r_1r_2 + 2br_2x_1 + \beta r_1 x_1 + \alpha r_2 y_1 - 2b\beta x_1^2 - \alpha\beta x_1 y_1$.

Let $\tau = 0$ in the characteristic equation (14), then we have

$$\lambda^2 - a\lambda + c = 0 \tag{15}$$

The roots of equation (15) are $\lambda_{\pm} = \frac{-a \pm \sqrt{b^2 - 4c}}{2}$.

Thus, equation (15) has negative real parts if and only if the following hypotheses hold:

Hypothesis 1. $a < 0$

Hypothesis 2. $c > 0$

Thus, the equilibrium point P is locally asymptotically stable if both conditions are satisfied $a < 0$, $c > 0$.

Suppose that $\tau \neq 0$ in the characteristic equation (14), if $\lambda = i\varpi$, $\varpi > 0$ is a root of the cahracterstic equation (14), then substitute the value of λ and the value of $e^{i\varpi\tau} = \left(cos(\varpi\tau) - i\, sin(\varpi\tau)\right)$ into equation (14), we get

$$-\varpi^2 - ai\varpi - b\, cos(\varpi\tau) - bi\, sin(\varpi\tau) + c = 0 \tag{16}$$

Separate the real and imaginary parts of equation (16), then $-\varpi^2 + c - b\, cos(\varpi\tau) = 0$ and $-a\varpi - b\, sin(\varpi\tau) = 0$.

which implies that,

$$-\varpi^2 + c = b\, cos(\varpi\tau), \text{ and } -a\varpi = b\, sin(\varpi\tau) \tag{17}$$

Square both sides of equations (17) respectively. Then, we get

$$\varpi^4 - 2\varpi^2 c + c^2 = b^2\, cos^2(\varpi\tau), \text{ and } a^2\varpi^2 = b^2\, sin^2(\varpi\tau) \tag{18}$$

Adding above equations (18) yields

$$\varpi^4 - (2c - a^2)\varpi^2 - b^2 + c^2 = 0 \tag{19}$$

The roots for equation (19) are

$$\varpi^2{}_{\pm} = \frac{(2c - a^2) \pm \sqrt{(2c - a^2)^2 - 4(-b^2 + c^2)}}{2}$$

Therefore, if the following hypothesis hold.

Hypothesis 3. $\{(2c - a^2) < 0\ and\ (-b^2 + c^2) > 0\}$ or $(2c - a^2)^2 < 4(-b^2 + c^2)$.

Thus equation (19) does not have any positive roots. Therefore, equation (14) does not have purely imaginary roots. As **Hypotheses 1** and **2,** ensure that all roots of equation (15) have negative real parts, it follows that, by Rouch´e's Theorem [18], the roots of equation (14) have negative real parts.

Thus, we have the following lemma, where the proof can be found in [19].

Lemma 1. If Hypotheses 1, 2 and 3 are satisfied then, all roots of equation (14) have negative real parts for all $\tau \geq 0$.

Also, if following hypothesis hold:

Hypothesis 4. $\{(2c - a^2) < 0\ and\ (-b^2 + c^2) > 0\}$ and $(2c - a^2)^2 = 4(-b^2 + c^2)$. Then equation (19) has a positive root $\bar{\omega}_{\pm}^2$.

On the other hand, if the following hypothesis satisfied:

Hypothesis 5. $\{(2c - a^2) < 0\ and\ (-b^2 + c^2) > 0\}$ and $(2c - a^2)^2 > 4(-b^2 + c^2)$. Then equation (19) has two positive root $\bar{\omega}_{\pm}^2$. In both cases equation (14) has purely imaginary roots when τ_k takes certain values.

Now, we are going to determine the values of τ_k as follows:

Dividing equations (17), yields

$$tan(\bar{\omega}\tau_k) = \frac{sin(\bar{\omega}\tau)}{cos(\bar{\omega}\tau)} = \frac{a\bar{\omega}}{\bar{\omega}^2 - c} \tag{20}$$

Apply tan^{-1} for both sides of equation (20), then

$$(\bar{\omega}\tau_k) = tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2 - c}\right) + 2k\pi, k = 0,1,2, \ldots \tag{21}$$

Equation (21) can be reduced to

$$\tau_k = \frac{1}{\bar{\omega}} tan^{-1}\left(\frac{(a_1(-\bar{\omega}^2 + a_4) + a_3 a_2)\bar{\omega}}{-\bar{\omega}^2 a_1 a_2 - \bar{\omega}^2 a_3 + a_3 a_4}\right) + 2k\pi\frac{1}{\bar{\omega}}, k = 0,1,2, \ldots \tag{22}$$

Substitute $\bar{\omega}_{\pm}^2$ into equation (22), then

$$\tau_k = \frac{1}{\bar{\omega}_{\pm}} tan^{-1}\left(\frac{a\bar{\omega}}{\bar{\omega}^2 - c}\right) + 2k\pi\frac{1}{\bar{\omega}_{\pm}}, k = 0,1,2, \ldots \tag{23}$$

The above discussion can be summarized into the following lemma, where the proof can be found in Cushing and Saleem [19].

Lemma 2. i) If Hypotheses 1, 2 and 4 are hold and $\tau = \tau_j^+$, then equation (14) has a pair of purely imaginary roots $\pm i\bar{\omega}_+$. ii) If Hypotheses 1, 2 and 5 are hold and $\tau = \tau_j^-$, then equation (14) has a pair of purely imaginary roots $\pm i\bar{\omega}_-$.

To find the necessary and sufficient conditions for non-existence of time delay induced instability, we introduce the following theorem:

Theorem 8. For an equilibrium point P^* to be asymptotically stable for all $\tau > 0$, if the following conditions are satisfied Kar [3]:

1) The real parts of all roots of equation $\Delta(\lambda, 0)$ are negative,
2) For all real $\bar{\omega}$ and $\tau \geq 0$, $\Delta(i\bar{\omega}, \tau) \neq 0$, where $i = \sqrt{-1}$

Now, we arrive to the following theorem.

Theorem 9. If the following conditions $a < 0$, $b < 0$ are hold, then the equilibrium point P of system (14) is locally asymptotically stable for all $\tau \geq 0$.

For the proof of the transversality conditions: $\frac{d}{d\tau} Re\lambda_j^+(\tau_j^+) > 0$, and $\frac{d}{d\tau} Re\lambda_j^-(\tau_j^-) < 0$.

We differentiate both sides of equation (14) with respect to τ

$$\lambda^2 - a\lambda - be^{-\lambda\tau} + c = 0$$

$$2\lambda\frac{d\lambda}{d\tau} - b\left(-\lambda e^{-\lambda\tau}\frac{d\lambda}{d\tau} - \tau e^{-\lambda\tau}\frac{d\lambda}{d\tau}\right) - a\frac{d\lambda}{d\tau} = 0$$

which implies that

$$\left(2\lambda + \tau be^{-\lambda\tau} - a\right)\frac{d\lambda}{d\tau} = b\lambda e^{-\lambda\tau} \quad (24)$$

Solve equation (24) for $\frac{d\lambda}{d\tau}$, we arrive at $\frac{d\lambda}{d\tau} = \frac{-b\lambda e^{-\lambda\tau}}{(2\lambda+\tau be^{-\lambda\tau}-a)}$.

But $\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{1}{\frac{d\lambda}{d\tau}}$, thus $\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-2\lambda-\tau be^{-\lambda\tau}+a}{b\lambda e^{-\lambda\tau}}$

which implies that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-2\lambda+a}{b\lambda e^{-\lambda\tau}} - \frac{\tau be^{-\lambda\tau}}{b\lambda e^{-\lambda\tau}} \quad (25)$$

Equation (25) can be simplified as

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-2\lambda+a}{b\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda} \quad (26)$$

From equation (14), we have

$$e^{-\lambda\tau} = \frac{\lambda^2-a\lambda+c}{b} \quad (27)$$

Substitute equation (27) into equation (26), yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{-2\lambda+a}{\lambda(\lambda^2-a\lambda+c)} - \frac{\tau}{\lambda}$$

We know that,

$$sign\left\{\frac{d(Re\lambda)}{d\tau}|_{\lambda=i\varpi}\right\} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}|_{\lambda=i\varpi}\right\}$$

Therefore,

$$sign\left\{\frac{d(Re\lambda)}{d\tau}|_{\lambda=i\varpi}\right\} = sign\left\{Re\left(\frac{-2\lambda+a}{\lambda(\lambda^2-a\lambda+c)}\right)|_{\lambda=i\varpi} - Re\left(\frac{\tau}{\lambda}\right)|_{\lambda=i\varpi}\right\} \quad (28)$$

Now, we evaluate the following to compute $sign\left\{\frac{d(Re\lambda)}{d\tau}|_{\lambda=i\varpi}\right\}$:

1) The value of $Re\left(\frac{-2\lambda+a}{\lambda(\lambda^2-a\lambda+c)}\right)|_{\lambda=i\varpi}$ is as follows:

$$Re\left(\frac{-2\lambda+a}{\lambda(\lambda^2-a\lambda+c)}\right)|_{\lambda=i\varpi} = Re\left(\frac{-2i\varpi+a}{i\varpi(\varpi^2-ai\varpi+c)}\right) = Re\left(\frac{-2i\varpi}{i\varpi(\varpi^2-ai\varpi+c)}\right) + Re\left(\frac{a_2}{i\varpi(\varpi^2-ai\varpi+c)}\right)$$

but,

$$Re\left(\frac{-2i\varpi}{-i\varpi(\varpi^2+ai\varpi-c)}\right)=Re\left(\frac{2}{\varpi^2-ai\varpi+c}\left(\frac{\varpi^2-ai\varpi-c}{\varpi^2-ai\varpi-c}\right)\right)=Re\left(2\left(\frac{\varpi^2-ai\varpi-c}{(\varpi^2-c)^2+a^2\varpi^2}\right)\right)=2\left(\frac{\varpi^2-c}{(\varpi^2-c)^2+a^2\varpi^2}\right)$$

and,

$$Re\left(\frac{a}{-i\varpi(\varpi^2-ai\varpi+c)}\right)=Re\left(\frac{a}{a\varpi^2+(-\varpi^2+c)i\varpi}\left(\frac{a\varpi^2-(-\varpi^2+c)i\varpi}{a\varpi^2-(-\varpi^2+c)i\varpi}\right)\right)=\frac{a^2}{a^2\varpi^2+(-\varpi^2+c)^2}$$

Therefore,

$$Re\left(\frac{2\lambda+a}{\lambda(\lambda^2-a\lambda+c)}\right)|_{\lambda=i\varpi}=\frac{2(\varpi^2-c)^2+a^2}{(\varpi^2+c)^2+a^2\varpi^2} \tag{29}$$

2) The value of $Re\left(\frac{\tau}{\lambda}\right)|_{\lambda=i\varpi}$ is as follows:

$$Re\left(\frac{\tau}{\lambda}\right)|_{\lambda=i\varpi}=Re\left(\frac{\tau}{i\varpi}\left(\frac{-i\varpi}{-i\varpi}\right)\right)=0 \tag{30}$$

Substitute equations (29) and (30) into equation (28), we arrive at

$$sign\left\{\frac{dRe(\lambda)}{d\tau}\right\}=sign\left\{\frac{2(\varpi^2-c)^2+a^2}{(\varpi^2+c)^2+a^2\varpi^2}\right\}=sign\{2(\varpi^2-c)+a^2\} \tag{31}$$

Theorem 10. Let $\tau_j^{\pm}$ be defined as in equation (23), if the following conditions:

i) $\lambda^2-a\lambda-be^{-\lambda\tau}+c=0.$

ii) $\{(2c-a^2)<0\ and\ (-b^2+c^2)>0\},\ and\ (2c-a^2)^2>4(-b^2+c^2)$ are both satisfied, then the equilibrium point P is stable when $\tau\in[0,\tau_0^+)\cup[\tau_0^-,\tau_1^+)\cup\ldots\cup[\tau_{m-1}^-,\tau_m^+)$ and unstable when $\tau\in[\tau_j^+,\tau_j^-)\cup[\tau_j^+,\tau_j^-)\cup\ldots\cup[\tau_{m-1}^+,\tau_{m-1}^-)$ for some positive integer m.

Proof. As the conditions in the theorem are satisfied, then we need only to verify the transversality conditions that are given by

$$\frac{dRe(\lambda)}{d\tau}|_{\lambda=\tau_j^+}>0\ and\ \frac{dRe(\lambda)}{d\tau}|_{\lambda=i\varpi^+}>0$$

$$\frac{dRe(\lambda)}{d\tau}|_{\lambda=\tau_j^-}>0\ and\ \frac{dRe(\lambda)}{d\tau}|_{\lambda=i\varpi^-}>0$$

From equation (19) and equation (31), it follows that:

From the conditions in Theorem 10, we have,

$$(2c-a^2)^2>4(-b^2+c^2)$$

Thus, $\sqrt{(2c-a^2)^2-4(-b^2+c^2)}>0$

which implies that $sign\left\{\sqrt{(2c-a^2)^2-4(-b^2+c^2)}\right\}>0.$

Therefore, $sign\left\{\frac{d(Re(\lambda))}{d\tau}|_{\lambda=i\bar{\omega}^{+},\tau=\tau^{+}}\right\} > 0.$

Now, $sign\left\{\frac{d(Re(\lambda))}{d\tau}|_{\lambda=i\bar{\omega}}\right\} = sign\{2(\bar{\omega}^2 - c)^2 + a^2\} = sign\left\{\sqrt{(2c-a^2)^2 - 4(-b^2+c^2)}\right\} > 0.$

From the conditions in the Theorem 10, we have,

$$(a_1^2 + 2a_4 - a_2^2)^2 > 4(-a_3^2 + a_4^2)$$

Thus, $$sign\left\{\sqrt{(2c-a^2)^2 - 4(-b^2+c^2)}\right\} > 0$$

which implies that $sign\left\{-\sqrt{(2c-a^2)^2 - 4(-b^2+c^2)}\right\} < 0$

Therefore, $$sign\left\{\frac{d(Re(\lambda))}{d\tau}|_{\lambda=i\bar{\omega}^{-},\tau=\tau^{-}}\right\} < 0$$

Thus, the transversability conditions are satisfied. This completes the proof.

Example 3. Consider system (12) with parameters $r_1 = 1, r_2 = 0.5$, $\alpha = 1$, $\beta = .05$, $b = 0.01$, and $H_x = 0$. The equilibrium point of system (12) is $(10, 0.9)$. For $\tau = 0$, the Jacobian matrix of system (12) at $(1.2, 1.52)$ has eigenvalues given by $\{-0.132 - 0.40617i, -0.132 + 0.40617i\}$. Note that the eigenvalues have negative real parts, thus the equilibrium point $(1.2, 1.52)$ is eigenvalues stable.

When $\tau \neq 0$ we have: $a = 4.4$, $b = -4.5$, and $c = -0.45$.

Now substitute the values of a, b,and c into equation (4:16), then we obtain: $\bar{\omega}_+^2 = 4.6017$ and $\bar{\omega}_-^2 = -4.40576$.

Take the square root of $\bar{\omega}_+^2$, then $\bar{\omega}_+ = \sqrt{4.8286} = 2.1974$.

Substitute the value of a, b, c and $\bar{\omega}_+$ into equation (23), then the value of the first time delay is given by: $\tau_0^+ = 0.16353$, and $\tau_1^+ = 1.5282$.

The critical value of time delay is $\tau = \tau_0^+ = 0.16353$. When $\tau < 0.16353$, the equilibrium point $(10, 0.9)$ is asymptotically stable, when $\tau = 0.16353$, the equilibrium point $(10, 0.9)$ may loss its stability, and when $\tau > 0.16353$, the equilibrium point $(10, 0.9)$ is unstable. The Figs. 7 and 8 clearly indicate that equilibrium points $(10, 0.9)$ are asymptotically stable when $\tau = \tau_0^+ = 0.16353$ and $\tau = \tau_1^+ = 1.5282$ respectively.

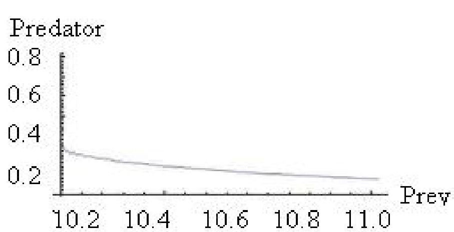

Fig. 7. Solution curve with $\tau_0^+ = 0.16\ 35\ 3$

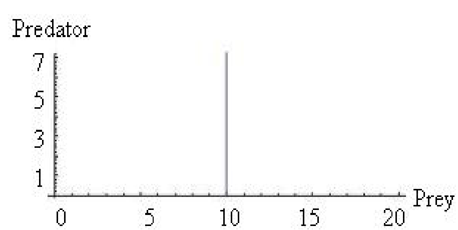

Fig. 8. Solution curve with $\tau_1^+ = 1.5\ 28\ 2$

5 Concluding Remark

Prey-predator models are of great interest to researchers in mathematics. Simple models such as the Lotka-Volterra are not able to tell us what is going on in the majority of cases. This can be attributed to the complexity of the biological ecosystem. Therefore, it is imperative to clearly understand the biological ecosystem and, hence, the urgency to develop a model.

The study was based on formulating a mathematical model to study the dynamics of the population densities of the prey-predator system. In the presented prey-predator models, the existence of the equilibrium points and the stabilities had been investigated. Depending on the values of the parameters, several possibilities were investigated. The equilibriums were obtained for a set of values of the parameters.

Finally, there is still a lot of work to do in prey-predator models with time delay and harvesting. The extension of the presented prey-predator models, which accommodates the effect of both delays and harvesting, is an ongoing challenge that stimulates more future research efforts.

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Competing Interests

Author has declared that no competing interests exist.

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